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Dario Ban

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Re-examination of centre of buoyancy curve and its evolute for rectangular cross section, Part 2: Using quadratic functions

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Summary

In this paper, exact hydrostatic particulars equations for the centre of buoyancy curve and metacentric locus curve are given for rectangular cross section using quadratic functions. Those equations have not been given for the hyperbola range of the heel angles so far, and here it is done by using basic quadratic functions and their horizontally symmetric immersion shapes, with two new methods defined: 1. Rotation of basic cross section shapes, and 2. Hydrostatic cross section area complement method that uses homothety or scaling properties of emerged and immersed areas of the rectangular cross section. Observed metacentric curve for rectangle consists of semi-cubic parabolas and Lamé curve with 2/3 exponent and negative sign, resulting in the cusp discontinuities in the symmetry of those functions definition. In order to achieve above, two theorems are given: the theorem about scaling using hydrostatic cross section area complement and the theorem about parallelism of centre of buoyancy tangents with waterlines. After non-dimensional bounds are given for the existence of the swallowtail discontinuity of metacentric curve for rectangular cross section in the Part 1 of this paper, the proof of its position in the symmetry of rectangle vertex angle is given in this Part 2 of the paper, thus confirming its position from theory.

Keywords: *centre of buoyancy curve; metacentric curve; rectangular cross section; quadratic functions; basic geometric shapes; hydrostatic area complement*

1 Introduction

1.1 Basic theory about metacentric locus curve

In the naval architecture theory and general theory of floating bodies, one of the most important hydrostatic properties for determination of ship's equilibrium is the centre of buoyancy and its evolute, i.e. metacentric locus M-curve. The foundations for their determination are set in the work of Archimedes, [1], Bouguer, [2], Euler, [3], Atwood, [3], La Croix, [5], and others, while their exact function equations are set in the work of many mathematicians like Huygens, [6], Neile, [7], and Van Heuraet, [8], with their examination of semi-cubical parabola as evolute of quadratic parabola. The usage of basic functions was first done by numerical integration methods by Kepler, [9], known by the name Simpson's rules, while D. W. Taylor, [9], first use polynomials for description of ship's geometry and her hydrostatic properties, directly. It was Bouguer in "Treate du Navire", [2], who already determined that centre of buoyancy curve is hyperbola, [11], and this was further examined in theory by many naval architects. In the 20th century, von Stainen, [12], and Robb, [13], have shown the possibility of description of centre of buoyancy curves for regular cross sections using quadratic functions, and their findings will be re-examined in this paper, with emphasis on hyperbola and its evolute Lamé curve with 2/3 exponent and negative sign, [14], [15].

Today, the modern theory about it is set in the bifurcation/catastrophe/singularity theory by Thom, [16], Zeeman, [16], Poston and Stewart, [18], and Arnold, [19], as a part of singularity theory, and control theory also. Recently, theoretical aspects of metacentric curve role in the evaluation of ship's stability are examined in the works of Mégel and Kliava, [20], Spyrou, [21], and many others, with the need for analytical examination of it for two-dimensional and three-dimensional problems for regular cross section shapes and bodies, like in recent quadratic approximation of centre of buoyancy curve for trapezoidal and pentagonal cross section in the work of Smirnov and Khashba, [22], or practical applications for general ship geometry and hydrostatics determination like in, [23], [24] and [25].

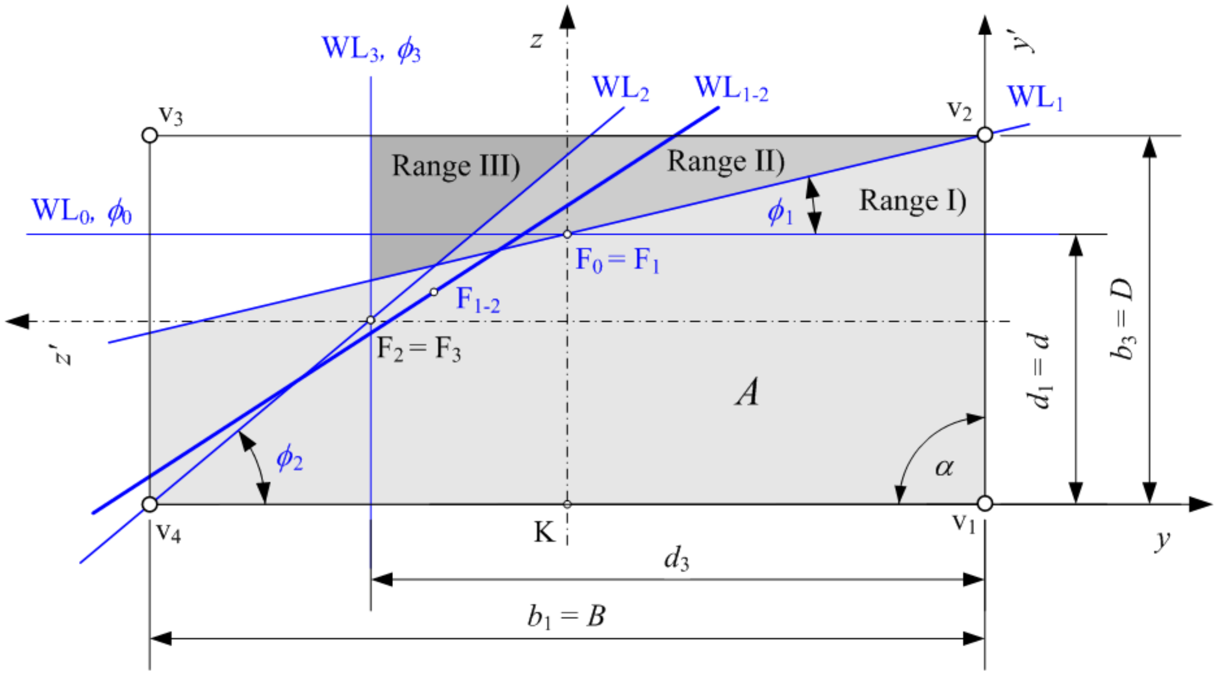

Fig. 1 Rectangle calculation regions defined by deck immersion/bottom emersion heel angle ranges

In general, regarding ship stability calculations, the goal is observing centre of buoyancy B-curve and its evolute M-curve for heel angles ϕ range from 0°to 90°. Therefore, in order to examine for heel angle values ϕ, three ranges I) to III) are determined for the first coordinate system quadrant, as shown in the Fig. 1, defined by geometry deck immersion/bottom emersion intersection with actual waterline for heel angles ϕ_1, ϕ_2 and 90°:

Range I) designated with light grey colour, for heel angles from $\phi_0 = 0°$ to the first deck immersion/bottom emersion heel angle ϕ_1, and waterlines from WL_0 to WL_1 with centre of floatation staying in a point $F_0 = F_1$. In this region, the quadratic equation that describes the centre of buoyancy B-curve is parabola that is proved in theory by many authors, like in Uršić, [26], and Von den Steinen, [12]. For this range, an initial calculation basic shape is rectangle with breadth $b_1 = B$ and draught $d_1 = d$, as shown in the Fig. 1, with initial immersed cross section area $A = Bd$.

Range II) designated with medium grey colour, for heel angles from the first deck immersion/bottom emersion heel angle ϕ_1 to the second deck immersion/bottom emersion heel angle ϕ_2, and waterlines $WL_{1\text{-}2}$ between WL_1 and WL_2 with variable centre of waterline $F_{1\text{-}2}$. In this region, the quadratic equation that describes the centre of buoyancy B-curve is hyperbola that is not proved in theory yet, and that is done in this paper.

Range III) designated with dark grey colour, for heel angles from the second deck immersion/bottom emersion heel angle ϕ_2 to the heel angle ϕ_3 = 90°, and waterlines from WL_2 to WL_3 with centre of floatation staying in a point $F_2 = F_3$. In this region, the quadratic equation that describes the centre of buoyancy B-curve is parabola that is proved in theory by many authors, also. For this range, an initial calculation basic shape is rectangle with breadth $b_3 = D$ and draught d_3 as shown in the Fig. 1, that will be calculated from the initial immersed rectangle cross section area $A = Bd$.

After the existence of the cusp discontinuities of the metacentric locus curve for rectangular cross section pontoon are examined in the Part 1 of the paper, [26], in this, Part 2 of the paper, the explicit equations of the centre of buoyancy and metacentric curve are given that enables additional examination of the position of the swallowtail cusp discontinuity in the hyperbola segment of heel angles, not done so far. Therefore, the equations of quadratic parabola, [28], and hyperbola, [29], and their evolutes, semi-cubical parabola, [6], [7], and Lamé curve with 2/3 exponent and negative sign, [14], [15], are given in this paper, both explicit and parametric forms of those equations, not done so far also.

The theory of general quadratic functions with their application for the description of basic geometrical shapes is then given in the Chapter 2, with the description of two new/old methods for centre of buoyancy B-curve and metacentric locus M-curve description: 1. Rotation of basic cross section shapes, and 2. Hydrostatic complementary cross section area scaling method. Both mentioned methods are based on the basic, horizontally symmetric, immersed cross section shapes of a pontoon and finding appropriate one for description using quadratic functions, with former already used in theory by Uršić, [26], and Von den Steinen, [12], for the parabola equation determination of the centre of buoyancy curve in the Range III) of the calculation heel angles ϕ, that is described in the Subchapter 3.4 of the paper. In this paper, the rotation of the isosceles triangle is used in the Range II) to determine hyperbola equation of the centre of buoyancy curve as will be explained further.

In the Chapter 3, the centre of buoyancy B-curve and metacentric M-curve, explicit and parametric equations are given, using above mentioned quadratic functions and their evolutes, for all three curve segments in the first coordinate system quadrant, not done so far for all segments. The overall equations for calculation of rectangular cross section centre of buoyancy B-curve and metacentric M-curve for all three heel angle segments using quadratic functions are then given in the Chapter 3.5 of the paper, for the situations of the existence of sixteen cusps, for $A > A_T/2$ and $A \leq A_T/2$, together with equations for semi-cubical parabola and cuspidal Lame curve as metacentric locus M-curve segments.

The evaluation of the obtained equations for the calculation of the centre of buoyancy B-curves and metacentric M-curves in the hyperbola segment of heel angles, between the first and the second deck immersion/bottom emersion angles, is then done in the Chapter 4, with the examples of the both cases of an immersion area of the rectangular shape relation to maximal cross section area, i.e. $A \leq A_T/2$ and $A > A_T/2$. For the first case $A \leq A_T/2$, the rotated triangular cross section method is applied, while for the second case, where $A \leq A_T/2$, the hydrostatic cross section area complement method is used, showing the accuracy of the methods set in this paper.

1.2 General immersion shapes for the first coordinate system quadrant

The main geometric characteristic of a pontoon with rectangular cross section geometry is that it has horizontal and vertical symmetry and four vertices v with all vertex angles equal

$\alpha = 90°$. Therefore, in order to examine the centre of buoyancy B-curve and metacentric M-curve characteristics, only one coordinate system quadrant can be examined, with three heel angle ϕ ranges that are observed in the first quadrant of coordinate system $y - z$, with the characteristics described in the Fig. 1, above.

As a results of a rectangular pontoon transversal inclination, the intersection of rectangular pontoon with waterplane for arbitrary heel angle ϕ results in the change of the immersed volume shape giving three basic geometric shapes depending on the general condition below.

Condition 1: The ratio of immersed and total rectangle cross section areas

$$A \leq A_T/2 \text{ or } A > A_T/2 \tag{1}$$

The basic shapes that can occur for rectangular cross section in the first coordinate system quadrant as shown in the Figs. 2 and 3, below, are then:

Case 1: General rectangular trapezoid,

Case 2: General rectangular triangle, for $A \leq A_T/2$

Case 3: General rectangular convex pentagon, for $A > A_T/2$

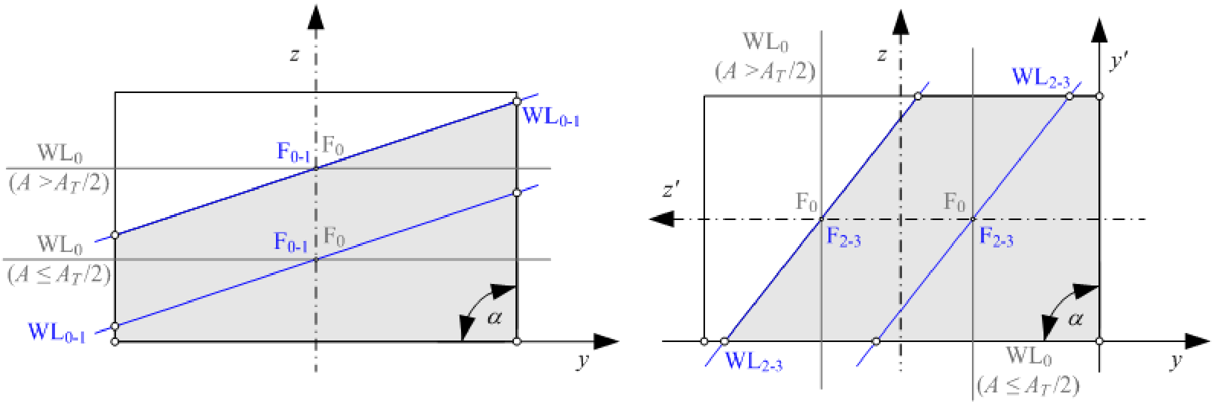

Fig. 2 General rectangular trapezoid cases

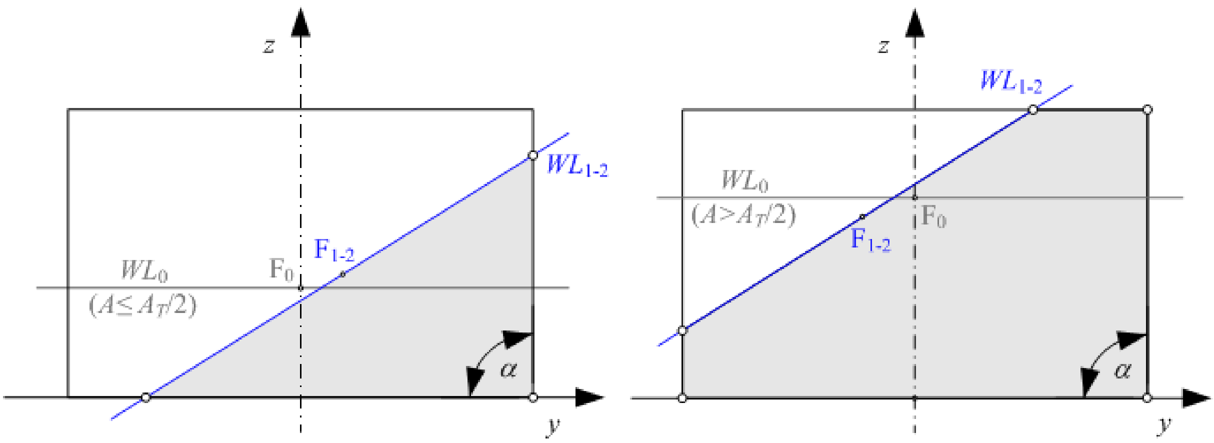

Fig. 3 a) General rectangular triangular cross section case, b) General rectangular pentagonal cross section case

The equation of centre of buoyancy B-curve in the first Range I) is quadratic parabola and for this heel angle range basic quadratic function can be used, directly. The same is for the Range III) of the heel angles but for rotated coordinate system axes for 90°, thus basic quadratic function can be used too, as for Range I), for general rectangular trapezoid immersion shapes shown in the Fig. 2, above.

For the rectangle heel angle Range II), the rectangular triangle immersion shape occur for $A \le A_T/2$ and rectangular pentagon immersion shape occur for $A < A_T/2$, as shown in the Fig. 3, above, for which suitable quadratic functions generating immersed shapes should be found, with corresponding basic quadratic functions description, as will be shown further in the paper.

2 General quadratic functions and their evolutes

2.1 Centre of buoyancy curves

The general equation for centre of buoyancy B-curve description for regular floating body cross sections with unit length until the first deck immersion/bottom emersion heel angle ϕ_1 of floating pontoon deck immersion/bottom emersion is given by Von den Steinen, [12], with quadratic function:

$$y^2 = 2r_0 z - \beta^2 z^2 \tag{2}$$

where β is function parameter, and $r_0 = b^2/a$, with b and a being horizontal and vertical semi-axis of quadratic functions.

This formula represents the explicit equation of possible plane intersections with cone giving observed quadratic functions. It is obtained from metacentric curve calculation of centres of buoyancy coordinate components for rectangular cross section pontoon with respective parametric equations in y and z coordinates for ϕ as

$$y(\phi) = r_0 \frac{\tan\phi}{\sqrt{1 - \tan^2\alpha \cdot \tan^2\phi}} \tag{3}$$

$$z(\phi) = \frac{r_0}{\tan^2\alpha}\left[1 - \frac{1}{\sqrt{1 - \tan^2\alpha \cdot \tan^2\phi}}\right] \tag{4}$$

where α is vertex angle of observed geometry as shown in the Fig. 1, above.
The formulae in (2) can be written as

$$y^2 = 2\frac{b^2}{a} z - \beta^2 z^2 \tag{5}$$

By introducing initial metacentric radius $\overline{\mathrm{M_0B_0}}$ (5) becomes

$$y^2 = 2\overline{\mathrm{M_0B_0}} z - \beta^2 z^2 = 2\frac{I}{\nabla} z - \beta^2 z^2 \tag{6}$$

where $r = \overline{\mathrm{M_0B_0}}$, $I(\phi)$ is moment of inertia of actual waterline, depending on angle of heel ϕ, and ∇ is constant volume displacement of the floating body.

Therefore, above equation (6) depends on parameter β with its values valid for respective basic cross section shapes that generate following quadratic equations that will be used for centre of buoyancy curve description in heel angle ranges I) to III):

$$\beta \equiv \begin{cases} \beta^2 > 1, \text{Ellipse with larger horizontal axis, for Ellipse} \\ \beta^2 = 1, \text{Circle, for Circle} \\ 0 < \beta^2 < 1, \text{Ellipse with larger vertical axis, for Ellipse} \\ \beta^2 = 0, \text{Parabola, for Rectangle} \\ \beta^2 < 1, \text{Hyperbola, for Isosceles Triangle} \end{cases} \tag{7}$$

Since the goal of this paper is examination of hydrostatic properties for rectangle, only, the equations for circle and ellipse cross sections are omitted here, and regular shapes with vertices will be examined further. Corresponding centre of buoyancy equations for rectangle, isosceles triangle and trapezoid, basic cross section geometries, are shown in Table 1, below.

Table 1 Centre of buoyancy curve equations for rectangle, isosceles triangle and trapezoid

Cross section geometry	Centre of Buoyancy Equation	Description
Rectangle	$y^2 = 2pz$	Parabola, $p = b^2/a$
Isosceles Triangle and Isosceles Trapezoid	$\frac{4b^2}{9}z^2 - a^2y^2 = \frac{4b^2a^2}{9}$	Hyperbola, $a = 2/3d$

It can be seen already that the quadratic function for the description of B-curve in the first Range I) of heel angles is quadratic parabola with equation given in the Table 1. The equations for other heel angle ranges should be derived yet and that will be done in the Chapter 3, with belonging parametric equations for heel angle ϕ, needed in naval architecture for practical calculation, given also.

2.2 Evolutes of centre of buoyancy curves

The metacentric curve characteristics are well studied in the hydrostatic theory of the floating bodies as well as in the mathematical singularity theory. This curve basically depends on centre of buoyancy B-curve characteristics, as it analytically represents metacentric locus or evolute M-curve of centre of buoyancy B-curve $z = f(y)$ of a floating body.

In the naval architecture, the relation for the determination of the metacentre M position for known centre of buoyancy B and for given heel angle ϕ, can be obtained using actual metacentric radius $r = \overline{\mathrm{MB}} = I/\nabla$ value with

$$y_{\mathrm{M}} = y_{\mathrm{B}} + I(\phi)/\nabla \sin\phi \tag{8}$$

$$z_{\mathrm{M}} = z_{\mathrm{B}} + I(\phi)/\nabla \cos\phi \tag{9}$$

Above parametric equations in (8) and (9) will be further used for metacentric M-curve determination later in the paper, while the character of the curves will be determined using explicit equations from the Table 2, below.

Table 2 Evolutes of centre of buoyancy curve equations for rectangle, isosceles triangle and trapezoid

Cross section geometry	Metacentric Curve Equation	Description
Rectangle	$y^2 = \frac{8}{27}\left(\frac{z^3}{p} - 3z^2 + 3pz - p^2\right)$	Semi-cubical Parabola $p = b^2/a$
Isosceles Triangle and Isosceles Trapezoid	$\left(\frac{2b}{3}z\right)^{\frac{2}{3}} - (ay)^{\frac{2}{3}} = \left(\frac{4b^2}{9} + a^2\right)^{\frac{2}{3}}$	Cuspidal Lamé curve $a = 2/3d$

The Tables 1 and 2, above, give explicit equations for centre of buoyancy B-curve and metacentric M-curves for rectangular and triangular basic cross shapes, as defined in theory in Von den Steinen, [12] and Uršić, [26]. For rectangular basic cross section, the centre of buoyancy B-curve is parabola, [28], with its evolute, the metacentric M-curve, being semi-cubical parabola, [30]; while for triangular basic cross section, the centre of buoyancy B-curve is hyperbola, [29], with its the metacentric M-curve being Lamé curve with exponent 2/3 and minus sign, [14], as shown in the part 1 of the paper, [26]. (This name for Lamé curve with exponent 2/3 and minus sign will be shortened in the text further on to cuspidal Lamé curve.) Both metacentric M-curve functions for rectangle and triangle basic shapes in the Table 2 are even functions, horizontally symmetrical, and therefore have cusp discontinuity in the symmetry of their definition, and therefore the same feature will be required for their definition. In the case of inclined triangles, that means that they should be isosceles triangles with intrinsic symmetry that they should have, as will be shown in the next chapters of the paper.

The most interesting part of the metacentric curve is between the first and the second deck immersion/bottom emersion angles with hyperbola part and here it will be determined using above set of basic quadratic functions.

2.3 Usage of basic quadratic functions for general immersed cross sections

After three heel angle ranges are set in the Fig. 1, the basic quadratic functions for the triangle and the rectangle can be applied for belonging immersed shapes generators defined for general rectangular trapezoid in I) and III) heel angle range and for general rectangular triangle for heel angle range II). But they cannot be applied for rectangular pentagon directly, for which different method should be found. Therefore, the methods for usage of quadratic functions for B-curve and metacentric M-curve description in general cases of rectangle immersion shapes are:

1. Direct method,
2. Rotated basic quadratic functions geometry shapes,
3. Scaling method using hydrostatic area complement.

Between above mentioned methods, the direct method is already implemented in theory for the Range I) as explained before, and represents the basics for the implementation for other two methods with rotated basic shapes and complementary ones. The second method is already applied for the Range III), also, where 90° rotated rectangle is applied, with breadth $b_3 = D$ and draught d_3, in order to determine hydrostatic characteristics in that heel angle range.

Detailed formulas for centre of buoyancy B-curve and metacentric M-curve can be then examined also, for all three heel angle ranges using above methods.

2.4 Rotated basic geometric shapes as quadratic functions generators

The basic set of conditions for some basic shape and its belonging quadratic function to be applicable for calculation of centre of buoyancy B-curve in general is:

Condition 2: The applicability of rotated basic geometry shape for quadratic function generation

1. Geometrical applicability,
2. Horizontal Symmetry (even functions),
3. Cross section area A' equal to initial one A_0:

$$A' = A_0 \tag{10}$$

If the set of the conditions above are satisfied, it is possible to find new basic shape dimensions breadth b' and draught d' on observed waterline WL, i.e. quadratic function semi-axes b' and a', necessary for their definition, as shown in Fig. 4 for general rectangular triangle for the case where $A \leq A_T/2$, below.

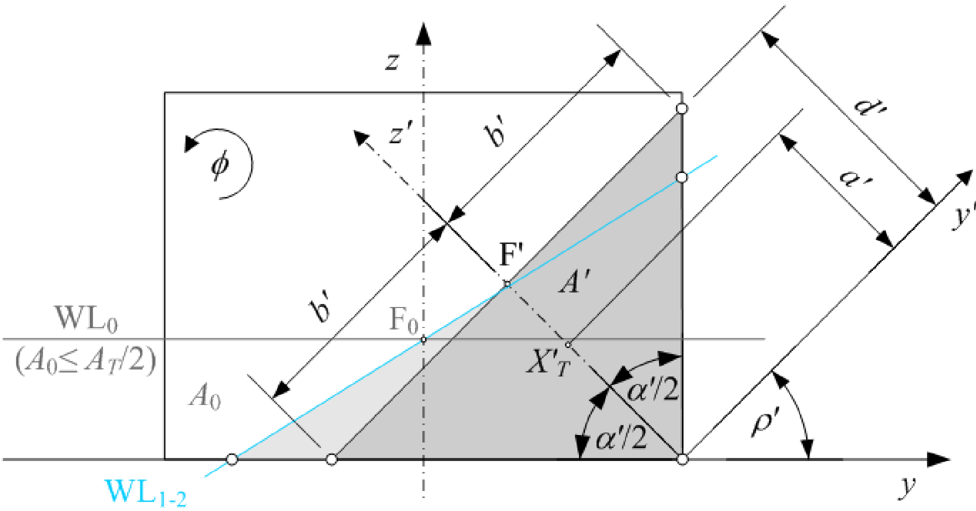

Fig. 4 Rotated isosceles triangle for hyperbola function definition

For that purpose, the new coordinate system $y' - z'$ should be set using affine transformations of the initial coordinate system $y - z$. Therefore, global coordinate system $y - z$ must be translated in one of the geometry vertices v with translation T and then rotated for the suitable angle ρ' using rotation R, first, to obtain new coordinate system $y' - z'$ for which basic geometrical shape can be built with the main, initial hydrostatic condition of having unit buoyancy or area A' equal to initial one A_0, as set in equation (10) in the *Condition 2*.

Then, the quadratic function $z' = f(y')$ for the basic cross section shape can be built using equations from the Table 1, above, with semi-axes determined from the immersed shape dimensions like shown in the Fig. 4.

Mathematically written, this method consists of affine transformations of coordinate system $y - z$ with rotation R for angle ρ' and translation T to the vertex v(y_v, z_v) of the observed geometry, with equation

$$\begin{Bmatrix} y \\ z \end{Bmatrix} = R_{\rho'} \cdot \begin{Bmatrix} y' \\ z' \end{Bmatrix} + T_v = \begin{bmatrix} \cos(\rho') & \sin(\rho') \\ -\sin(\rho') & \cos(\rho') \end{bmatrix} \begin{Bmatrix} y' \\ z' \end{Bmatrix} - \begin{Bmatrix} y_v \\ z_v \end{Bmatrix} \tag{11}$$

Obtained equation $z' = f(y')$ must be then rotated back to initial coordinate system with the rotation for opposite angle $-\rho'$ and reverse translation $-T$. I.e., it is necessary to rotate obtained equations for $-\rho'$ angle and translate them to the origin by translation T($-y_v$, $-z_v$) to obtain results in the initial coordinate system $y - z$, with:

$$\begin{Bmatrix} y \\ z \end{Bmatrix} = R_{-\rho'} \cdot \begin{Bmatrix} y' \\ z' \end{Bmatrix} - T_v = \begin{bmatrix} \cos(-\rho') & \sin(-\rho') \\ -\sin(-\rho') & \cos(-\rho') \end{bmatrix} \begin{Bmatrix} y' \\ z' \end{Bmatrix} + \begin{Bmatrix} y_v \\ z_v \end{Bmatrix} \tag{12}$$

Now, the equation for the centre of buoyancy B-curve can be given for observed rectangular cross section for the heel angle Range IIa) in the Subchapter 3.2., with example of calculation given in the Subchapter 4.1.

Except the B-curve, the equation for metacentric locus M-curve can be given also, according to the theoretical examination in the Subchapter 2.2, above, as will be shown for hyperbola, below. This procedure is already used in theory for determination of rotated parabola for Range III) of the heel angles, as shown in the works of Robb, [13], and textbook from Uršić, [26].

2.4.1 Hyperbola

In the Range IIa), for heel angles between the first and the second deck immersion/bottom emersion heel angles ϕ_1 and ϕ_2, where *Condition 1* $A \leq A_T/2$ case is valid, the basic shape that has horizontal symmetry is isosceles triangular shape, with symmetry line in the middle of vertex angle $\alpha = 90°$ at $\alpha' = 45°$, as shown in Fig. 4, above. Therefore, new rotated coordinate system $y' - z'$ is set to lower right rectangle vertex v_1, with rotation angle ρ'. The equation for the centre of buoyancy B-curve in this heel angle region, Range IIa), is then hyperbola, as it is defined in theory.

In order to give hyperbola equation here, it is needed to determine its belonging semi-axis b' and a' that can be determined from an isosceles triangle breadth b' and height d' as shown in the Fig. 4, above. From the last, the third item of the basic shape applicability in the *Condition 2*, the dimensions of the isosceles triangle can be determined using triangle area equation with $A = A_0 = Bd_0 = b'd'$ and vertex angle $\alpha' = \alpha/2$ as

$$d' = \sqrt{\frac{A}{\tan\alpha'}} = \sqrt{\frac{Bd_0}{\tan\alpha/2}} \tag{13}$$

$$b' = d'\tan\alpha' = d'\tan\alpha/2 = \sqrt{Bd_0\tan\alpha/2} \tag{14}$$

The horizontal hyperbola semi-axis b' is then already determined, while the vertical semi-axis a' can be calculated from the fact that it equals two thirds of triangle height with

$$a' = 2/3d' = 2/3\sqrt{Bd_0/\tan\alpha/2} \tag{15}$$

By knowing hyperbola semi-axes, the centre of buoyancy B-curve equation can be determined for general rectangular triangle case of immersed shape with

$$\frac{4b'^2}{9}z'^2 - a'^2y'^2 = \frac{4b'^2a'^2}{9} \tag{16}$$

The direct hyperbola equation for Range IIa) will be given after in the Subchapter 3.2.

Its belonging metacentric M-curve is cuspidal Lamé curve that has extreme E cusp discontinuity in this Range IIa) at $\phi_{E1} = 45°$, with properties that will be investigated further in the paper in order to prove the *Definition 2* from the Part 1 of the paper for extreme E cusp discontinuity position on the line $z = \pm y$, from Zeeman, [16]. But it is obvious from Fig. 4 that hyperbola extremes can occur on somewhat different line, i.e. the line does not go through the coordinate system origin. Therefore, the correction can be set for a new coordinate system position $y' - z'$ as shown in the Fig. 4, above.

The correction of the definition for the discontinuity position is then:

Definition 1: The extremes of hyperbola segments for rectangular cross section occur on the lines

$$z = \pm y \mp (B/2 - D) \tag{17}$$

It is still needed to give exact B-curve and M-curve equations, and that will be done in the Chapter 3 of this paper. And, it is needed to check the *Definition 1* correction in the Range IIb), also, with examination in the following subchapter.

2.5 Hydrostatic cross section area complement

In the case of pentagonal immersed cross section area geometry, Fig. 3, detected for Range IIb) where *Condition 1* $A > A_T/2$ case is valid, there is no direct basic geometry to be applicable for centre of buoyancy B-curve and metacentric locus M-curve quadratic function determination. In order to find them, novel hydrostatic cross section area complement method is applied here based on scaling of emerged part of total cross section area, where complementary area belongs to basic ones for quadratic function determination from Table 1, with explanation of the method below.

The hydrostatic area complement method uses basic property of scaled geometry shape preservation and performs its transformation by scaling all components of observed geometry around point X_T, by some ratio k that is equal to the ratio of an initial and complement area, A and $\overline{A}$. That means that the length and the areal particulars of observed complementary geometry can be scaled around point X_T for above mentioned scale ratio k, directly.

This hydrostatic complement method mathematically represents geometrical shape scaling or mapping around a point, i.e. homothety with scale coefficient $k < 0$. Homothety in the mathematical terms is a transformation of an affine space determined by a point X_T called its *centre* and a nonzero number k called its *ratio*, with the main characteristic of preserving mapped geometry shape during transformation.

The ratio k can be easily determined in the hydrostatic complement method using the ratio of initial, immersed and complementary, emerged cross section area as:

$$k = -\overline{A}/A' = \overline{A}/A \tag{18}$$

I.e.:

$$k = -(A_T - A)/A \tag{19}$$

where A is initial cross section area and $\overline{A} = (A_T - A)$ is its complementary area.

Since, the main goal of this method is determining the centre of buoyancy B-curve and metacentric M-curve of initial, immersed geometry using quadratic functions, the semi-axes b and a for their definition has to be calculated, and here it is done by knowing semi-axes $\overline{b}$ and $\overline{a}$ of centre of buoyancy $\overline{\mathrm{B}}$ -curve of the complementary cross section and their scaling for scale ratio k, determined above in the equation (19). The theorem about above can be given here with:

Theorem 1: Scaling using hydrostatic cross section area complement

For defined cross section geometry, the area A immersed in some fluid has locally horizontally symmetric hydrostatic area complement $\overline{A}$ such that its belonging centre of buoyancy B-curve and metacentric locus M-curve, as well as hydrostatic complement area geometry, can be scaled for the ratio $k = -A/\overline{A}$, around scaling point X_T representing the centroid of the defined cross section geometry.

The theorem above can be written for some general geometry feature G as

$$G' = |k| \cdot G \tag{20}$$

where geometry G features are length and areal particulars, length l, centroid C, area A, and so on, and G' is a scaled shape.

For the situation of complementary geometry above, the ratio k is negative, and it is called reverse mapping with

$$k < 0 \tag{21}$$

If equation (20) is written in vector form one gets

$$\mathbf{P}' = X_T - k(\mathbf{P} - X_T) \tag{22}$$

where **P** is initial point and **P′** is a point after scaling.

In the naval architecture practice, usual stability calculations are done for transversal or longitudinal inclination angles, and therefore required hyperbola equations will be given in parametric form for heel angles, and heel angle ϕ will be the main input variable for centre of buoyancy B-curve and metacentric M-curve calculation, as will be shown further in the paper.

2.5.1 Geometric settings

Since general immersion shapes for rectangle, shown in the Fig. 3, contain rectangular pentagon also, there is no direct usage of quadratic formula and their basic shapes for description of belonging B-curve and M-curve. But, there is a yet another possibility of their usage by building horizontally symmetric isosceles triangle as hydrostatic cross section area complement to initial general rectangular pentagon, as shown in the Fig. 5, below.

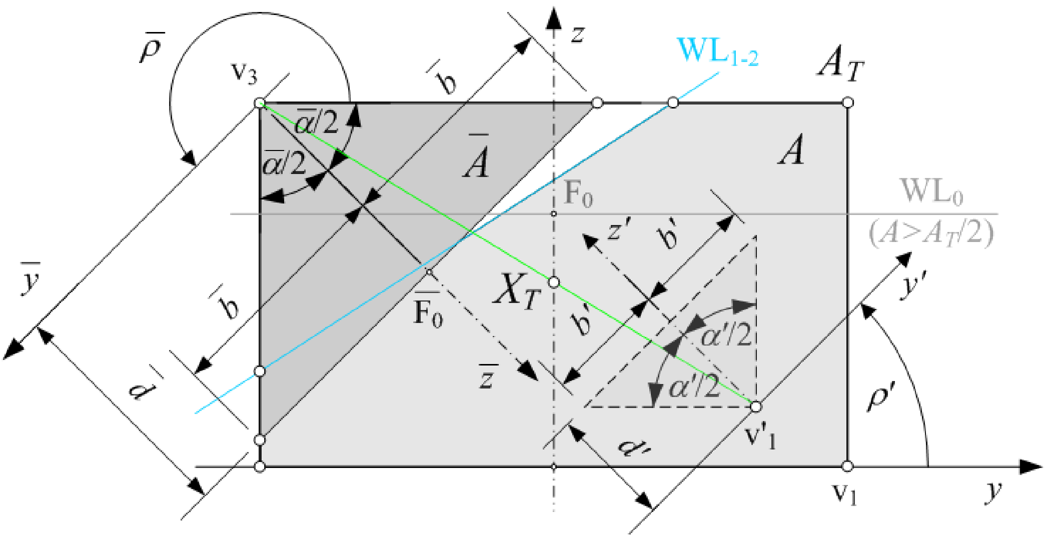

Fig. 5 Isosceles triangle as hydrostatic cross section area complement of rectangular pentagon, with its scaling

The basic set of conditions for some basic shape and its belonging quadratic function to be applicable as complementary to some general cross section is:

Condition 3: The applicability of hydrostatic area complement

1. Geometrical applicability,
2. Horizontal Symmetry (even functions),
3. Complementary cross section area A_C to initial A_0 one, i.e.:

$$A_C = A_T - A_0 \tag{23}$$

where A_T is total cross sectional area of observed geometry.

In hydrostatic based terminology above can be set as

$$A_E = A_T - A_I \tag{24}$$

where: A_T is total cross section area of observed pontoon shape, A_E is emerged part of observed pontoon shape representing complementary area $A_C = A_E$, and A_I is immersed part of total cross section area A_T, that creates hydrostatic buoyancy force B opposite to the weight force W.

Since, initial cross section area A_I is always constant in hydrostatic field, i.e. holonomic constraint exists there, therefore it is $A_I = A_0 = A =$ const., as set in (10).

But, the hyperbola B-curve equation for the isosceles triangle is known for the coordinate system $y' - z'$ set to rectangle's vertex v_3 angle α symmetry, as shown on Fig. 5, above, and therefore above equation (23) could be written as

$$\overline{A} = A_T - A' \tag{25}$$

where $\overline{A} = A_C$ and $A' = A_0$ for coordinate system rotated for angles $\overline{\rho}$ and ρ', respectively.

If above basic three parts of *Condition 3* are satisfied, it is possible to find new basic shape dimensions breadth $\overline{b}$ and draught $\overline{d}$ on observed waterline WL, i.e. quadratic function semi-axes $\overline{b}$ and $\overline{a}$, as shown in Fig. 5, above, for the isosceles triangle that is the complement of initial pentagonal immersion shape of rectangle, necessary for the definition of hyperbola.

The basic property of geometric area complement regarding its relation between rotations of their coordinate systems $y' - z'$ and $\overline{y} - \overline{z}$ for reverse mapping with $k < 0$ is then

$$\overline{\rho} = 180 + \rho' \tag{26}$$

By knowing hyperbola equation $\overline{\mathrm{H}}$ for the complementary triangle, it is possible to determine required centre of buoyancy B-curve equation H, by using direct scaling around a centroid point method, with equations explained in the text below.

The only input data needed for above Hydrostatic cross section complement scaling method are initial geometry data about its area A_T and its centroid X_T, initial cross section draught d_0 and immersed area A_0, and belonging centre of buoyancy B-curve equation, preferably being basic quadratic function. The homothetic ratio k can be then determined by ratio $-A/\overline{A}$, and scaling of all complement geometry particulars done around scaling centre point X_T, as will be shown in the next subchapter and in the Fig. 8, below.

Since semi-axes are needed for the description of the centre of buoyancy B-curve of immersed area using quadratic functions, only, they can be calculated easily using equation (20) and the equation for ratio k from (19) as

$$b' = |k| \cdot \overline{b} \tag{27}$$

$$a' = |k| \cdot \overline{a} \tag{28}$$

It is still needed to scale the original rectangle vertex v_1 to the point v'_1, as shown in the Fig. 5, above, by green projection line, together with scaling of two rectangle intersection points with WL, $P_1(y_1, z_1)$ and $P_2(y_2, z_2)$, as shown in the Fig. 8 by using scaling equation in the vector form from (22).

The example of the calculation using this method will be shown in the Subchapter 4.1, below.

2.5.2 Hyperbola

In the Range IIb), for heel angles between the first and the second deck immersion/bottom emersion heel angles ϕ_1 and ϕ_2, where *Condition 1* is valid with $A > A_T/2$, there is no basic shape that can be built for general rectangular pentagon case of immersed initial triangular shape. But, complementary isosceles triangle shape can be built in the opposite rectangle vertex v_3, with symmetry line in the middle of vertex angle α = 90° at $\bar{\alpha}/2$ = 45°, as shown in the Fig. 5, above. Therefore, the new rotated coordinate system $\bar{y} - \bar{z}$ with its complementary $y' - z'$ coordinate system is obtained in the lower right rectangle vertex v_1 with rotation angle ρ'. The equation for the centre of buoyancy B-curve in this heel angle region, Range IIb) is then complementary hyperbola as mentioned in theory. In order to determine its equation, the semi-axes of complementary isosceles triangle $\bar{b}$ and $\bar{a}$ should be determined first similar to rotated basic immersion shape in the Subchapter 2.4.1, using equations (13) to (15) one gets

$$\bar{d} = \sqrt{\frac{\bar{A}}{\tan\alpha'}} = \sqrt{\frac{(A_T - A_0)}{\tan\alpha'}} = \sqrt{\frac{(BD - Bd_0)}{\tan\alpha/2}} = \sqrt{\frac{B(D - d_0)}{\tan\alpha/2}} \tag{29}$$

$$\bar{b} = \bar{d}\tan\alpha/2 = \sqrt{(BD - Bd_0)\tan\alpha/2} = \sqrt{B(D - d_0)\tan\alpha/2} \tag{30}$$

$$\bar{a} = 2/3\bar{d} = 2/3\sqrt{B(D - d_0)/\tan\alpha/2} \tag{31}$$

By knowing complementary hyperbola semi-axes $\bar{b}$ and $\bar{a}$, required semi-axes b' and a' can be the calculated too, using direct Hydrostatic complementary cross section area scaling method as shown above. Thus, for the horizontal hyperbola semi-axis b' and the vertical semi-axis a' calculated for scaled isosceles triangle with origin in the point v'_1($y_{v'_1}$, $z_{v'_1}$), the exact formulas of B and M-curve equations can be given as shown in the following Chapter 3.

Similar to Range IIa), the extremes of hyperbola segments for rectangular cross section in Range IIb) can occur on the lines $z = \pm y \mp (B/2 - D)$ as in equation (17), or $z' = \pm y'$, thus proving the *Definition 2* from the Part 1 of the paper, [26].

3 Centre of buoyancy and metacentric curve characteristics

In this chapter, the centre of buoyancy and metacentric curve equations are given for all three heel angle ranges from I) to III) using quadratic functions and methods defined in the Chapter 2. Beside above basic situations for heel angle values until the first deck immersion/bottom emersion angle ϕ_1 and trapezoid cross sections, in Range I), there are other geometrical cases, as explained in Chapter 1 and Fig. 3, for other heel angle ranges that should be solved, too. For that purpose, the usage of the basic quadratic functions are then showed in the following subchapters, bellow.

3.1 Parabola segment, Range I)

According to the hydrostatic theory of ships in the works of Von den Steinen, [12], Robb, [13], and shown in the textbook of Uršić, [26], the type of curve that one obtains for the centre of buoyancy B-curve for a rectangular cross section of the pontoon, with breadth $b_1 = B$ and draught $d_1 = d_0 = d$, is a quadratic parabola in general form from Table 1. For the Range I), the parabola formula in coordinate system set in initial centre of buoyancy point B_0 is then

$$y^2 = 2p_1 z \tag{32}$$

where p_1 is the parameter of parabola, equalling equation for metacentric radius $\overline{\mathrm{M_0B}}$.
I.e.:

$$p_1 = r_1 = \overline{\mathrm{M_0B}} = I_0/\nabla = LB^3/(12LBd) = B^2/(12d) \tag{33}$$

where I_0 is initial waterline WL moment of inertia, and ∇ is volume displacement of the ship.

In required parametric form, the parabola equation of centre of buoyancy B-curve is then known from [26] with

$$y_{\mathrm{B}} = B^2/(12d)\tan\phi \qquad (34)$$

$$z_{\mathrm{B}} = B^2/(24d)\tan^2\phi + z_{\mathrm{B0}} \qquad (35)$$

3.1.1 Metacentric curve

After rewriting the cubic terms in Table 2 for the rectangular cross section metacentric M-curve part for heel angle ϕ range between 0° and the first deck immersion/bottom emersion heel angle ϕ_1, one gets semi-cubical parabola equation from [30], for coordinate system set in initial centre of buoyancy point B_0, with parameter p_1 as

$$p_1 y^2 = 8/27(z - p_1)^3 \qquad (36)$$

Rewritten for y one gets

$$y^2 = 8/27\left(z / p_1 - 3z^2 + 3p_1 z - p_1^2\right) \qquad (37)$$

By introducing the parabola parameter p_1 equation from (33), the equation of the semi-cubical parabola becomes

$$y^2 = 8/27\left(12d\,z^3/B^2 - 3z^2 + 3z\,B^2/(12d) - B^4/(12d)^2\right) \qquad (38)$$

Rewriting the above, finally, the equation for metacentric curve of rectangular pontoon cross section until the first deck immersion/bottom emersion heel angle ϕ_1 can be obtained as

$$y_{\mathrm{M}}^2 = 32d/(9B^2)z_{\mathrm{M}}^3 - 8/9\,z_{\mathrm{M}}^2 + 2B^2/(27d)z_{\mathrm{M}} - B^4/(243\cdot 2d)^2 \qquad (39)$$

Above formulas are valid for perpendicular sides of the rectangle cross section for heel angles until the first deck immersion/bottom emersion heel angle ϕ_1, i.e. immersion or emersion angle of the rectangular pontoon sides, where waterline length is influenced by cross section boundary.

3.2 Hyperbola segment, Range IIa)

In the range between the first and the second deck immersion/bottom emersion heel angles ϕ_1 and ϕ_2, triangular cross section shapes occur for rectangle intersection with waterline for $A_0 \leq A_T/2$ condition, as shown in the Fig. 4. The procedure for determination of hydrostatic curves for this cross section case is described before in the Subchapter 2.4.1 with explanation for hyperbolas. It can be seen in the Fig. 4, that for arbitrary waterline position $\mathrm{WL}_{1\text{-}2}$, the isosceles triangle satisfying three parts of the *Condition 2* can be set in the lower right vertex of the rectangle v_1. I.e., the triangle is geometrically applicable, symmetric and therefore isosceles, and has the same area A' as initial A_0, as defined in (10).

For semi-axes b' and a' calculated in the equations (14) and (15), the hyperbola equation from Table 1 in rotated coordinate system $y' - z'$ can be written as:

$$\frac{A\tan\alpha/2}{9}z'^2 - \frac{4A}{9\tan\alpha/2}y'^2 = \frac{4/9A\tan\alpha/2\,A/\tan\alpha/2}{9}$$

$$\frac{Bd_0\tan\alpha/2}{9}z'^2 - \frac{4Bd_0}{9\tan\alpha/2}y'^2 = \frac{4/9Bd_0\tan\alpha/2\,Bd_0/\tan\alpha/2}{9}$$

$$\left(Bd_0 \tan^2 \alpha/2\right) z'^2 - 4Bd_0 y'^2 = \frac{4}{9} B^2 d_0^2 \tan \alpha/2 \tag{40}$$

where all values are known and therefore, hyperbola equation is determined.

Belonging parametric form of above hyperbola equation for heel angles ϕ in (40) is

$$z' = a'/\cos(\phi') \tag{41}$$

$$y' = 2/3 b' \tan(\phi') \tag{42}$$

where parameter ϕ' is shifted $\alpha/2$ for symmetry of vertex angle α, i.e. c.s. $y' - z'$.

After introducing semi-axes b' and a' above parametric equations can be written as

$$z'_\mathrm{B} = 2/3\sqrt{Bd_0 / \tan \alpha/2} / \cos \phi' \tag{43}$$

$$y'_\mathrm{B} = 2/3\sqrt{Bd_0 \tan \alpha/2} \tan \phi' \tag{44}$$

And hyperbola equation for heel angles Range IIa) is given in this way.

It can be observed here that hyperbola equation can be obtained by knowing just two other parameters, isosceles triangle area A and the vertex angle α, instead of semi-axes. And that is another way of defining those equations.

3.3 Hyperbola segment, Range IIb)

In the Range IIb), it can be seen in the Fig. 5 that for arbitrary waterline position $WL_{1\text{-}2}$, the complementary isosceles triangle satisfying three parts of the *Condition 3* can be set in the upper left vertex of the rectangle v_3. I.e., the triangle is geometrically applicable, symmetric and therefore isosceles, and has the same area A' as initial A_0, as defined in (10).

By knowing complementary hyperbola semi-axes $\bar{b}$ and $\bar{a}$, the centre of buoyancy curve equation for the complementary triangle can be written as

$$\frac{4\bar{b}^2}{9}\bar{z}^2 - \bar{a}^2\bar{y}^2 = \frac{4\bar{b}^2\bar{a}^2}{9} \tag{45}$$

The easiest way for determining initial, required centre of buoyancy B-curve for the Range IIb) can be then done by using of Hydrostatic complement cross section area scaling method, described in the Chapter 3, by using scaling about centre point X_T by ratio k. By using the equations for b' and a' from (27) and (28), for the *Condition 1* where $A/A_T > 1/2$, in (1), the final equation for the initial centre of buoyancy B-curve then becomes

$$\frac{4k^2\bar{b}^2}{9} z'^2 - k^2\bar{a}^2 y'^2 = \frac{4k^4\bar{b}^2\bar{a}^2}{9} \tag{46}$$

Written for b' and a' above equation becomes

$$\frac{4b'^2}{9} z'^2 - y'^2 = \frac{4b'^2 a'^2}{9}$$

and it is the same equation as the equation (16) for hyperbola in the Range IIa).

The difference for those equations is in the coordinate system origins, where the origin of the equation (16) is in the rectangle vertex v_1 and for above the origin is translated in the point v'_1 around the centre point X_T.

Belonging parametric form of above hyperbola equation for heel angles ϕ' in (40) is the same as for Range IIa) with equations (41) and (42) from previous subchapter.

After introducing semi-axes b' and a', above parametric equations can be written as

$$z' = 2/3k\sqrt{Bd_0/\tan\alpha/2}\Big/\cos\phi',\; y' = 2/3\sqrt{Bd_0\tan\alpha/2}\tan\phi'$$

$$z' = \frac{2}{3}\frac{A_T - A}{A}\sqrt{Bd_0/\tan\alpha/2}/\cos\phi' = \frac{2}{3}\frac{BD - Bd_0}{Bd_0}\sqrt{\frac{Bd_0}{\tan\alpha/2}}/\cos\phi' \tag{47}$$

$$y' = \frac{2}{3}\frac{A_T - A}{A}\sqrt{Bd_0\tan\alpha/2}\tan\phi' = \frac{2}{3}\frac{BD - Bd_0}{Bd_0}\sqrt{Bd_0\tan\alpha/2}\tan(\phi') \tag{48}$$

and hyperbola equation for heel angles Range IIb) is given in this way, also.

In order to obtain formulas for global $y - z$ coordinate system, above hyperbola formulas should be translated and rotated to $y - z$ coordinate system origin using equation (12). In this way, all equations for Range II) are given for respective hyperbolas.

3.4 Parabola segment, Range III)

3.4.1 Centre of buoyancy curve

As explained in the Subchapter 2.1, the formula (39) is valid for the Range III) too, for rotated rectangle with breadth $b_3 = D$ and draught d_3, where y and z change places for angles from ϕ_2 to $\phi_3 = 90°$, as shown in the Fig. 1 and Fig. 2 for $y' - z'$ coordinate system, with corresponding parabola equation

$$z'^2 = 2p_3 y' \tag{49}$$

with parameter p_3 of corresponding parabola for Range III) equal to

$$p_3 = r_3 = \overline{\mathrm{MB}} = I/\nabla = LD^3/(12LDd_3) = D^2/(12d_3) \tag{50}$$

For rectangular pontoon of volume ∇ and length of the pontoon L set to 1, the value of the pontoon draught d_3, for heel angles between ϕ_2 and 90°, is equal $d_3 = Bd/D$, $d = d_0$.

The parameter p_3 of parabola (50) is then

$$p_3 = D^3/(12Bd) \tag{51}$$

The parabola equations in required parametric form with parameter p_3 from [26] are then:

$y'_B = D^2/(12d_3)\tan(90 - \phi)$

$z'_B = D^2/(24d_3)\tan^2(90 - \phi)$

$$y'_B = D^3/(12Bd)\tan(90 - \phi) \tag{52}$$

$$z'_B = D^3/(24Bd)\tan^2(90 - \phi) \tag{53}$$

Finally, the parametric parabola equations for the Range III), according to [26], in global, rectangular coordinate system, with corrected formulas are

$$y_B = \frac{B}{2} - \frac{Bd}{2D} - \frac{D^3}{24Bd}\tan^2(90 - \phi) \tag{54}$$

$$z_B = \frac{D}{2} - \frac{D^3}{12Bd}\tan(90 - \phi) \tag{55}$$

and thus, all the centre of buoyancy curve equations are obtained.

3.4.2 Metacentric curve

The equation of metacentric curve for Range III) according to semi-cubic parabola equation in $y'-z'$ coordinate system is

$$z'^2 = 8/27\left(y'^3/p_3 - 3y'^2 + 3p_3 y' - p_3^2\right) \tag{56}$$

After introducing parabola parameter p_3 one finally gets

$$z_{\mathrm{M}}'^2 = 32Bd/\left(9D^3\right)y_{\mathrm{M}}'^3 - 8/9\, y_{\mathrm{M}}'^2 + 2D^3/\left(27Bd\right)y_{\mathrm{M}}' - D^6/\left(243\cdot 2Bd\right)^2 \tag{57}$$

and thus, all the metacentric curve equations are obtained.

With above, all centre of buoyancy B-curve segment equations with their belonging metacentric locus M-curve equations are given directly. Thus, one of the main goals of the paper is fulfilled, and all equations will be given together in the next subchapter.

3.5 Overall equations for the first coordinate system quadrant

In this subchapter, all equations determined before for three heel angle ranges, in the first coordinate system quadrant, will be summarized in the tables below, for centre of buoyancy and metacentric curve equations.

First, the Table 3, below, shows parametric centre of buoyancy curve equations for all three heel angle ranges in the first coordinate system quadrant.

Table 3 Parametric centre of buoyancy curve equations for rectangle

Range	Heel Angles	Centre of Buoyancy Curve	
I)	$0° \le \phi < \phi_1$	$y_{\mathrm{B}} = B^2/\left(12d\right)\tan\phi$	$z_{\mathrm{B}} = B^2/\left(24d\right)\tan^2\phi + z_{\mathrm{B0}}$
IIa)	$\phi_1 \le \phi < \phi_2$ $A \le A_T/2$	$y_{\mathrm{B}}' = 2/3\sqrt{Bd_0\tan\alpha/2}\tan\phi'$	$z_{\mathrm{B}}' = 2/3\sqrt{Bd_0/\tan\alpha/2}/\cos\phi'$
IIb)	$\phi_1 \le \phi < \phi_2$ $A > A_T/2$	$y_{\mathrm{B}}' = \dfrac{2}{3}\dfrac{BD-Bd_0}{Bd_0}\sqrt{Bd_0\tan\dfrac{\alpha}{2}}\tan\phi'$	$z_{\mathrm{B}}' = \dfrac{2}{3}\dfrac{BD-Bd_0}{Bd_0}\sqrt{\dfrac{Bd_0}{\tan\alpha/2}}\dfrac{1}{\cos\phi'}$
III)	$\phi_2 \le \phi \le 90°$	$y_{\mathrm{B}} = \dfrac{B}{2} - \dfrac{Bd_0}{2D} - \dfrac{D^3}{24Bd_0}\tan^2(90-\phi)$	$z_{\mathrm{B}} = \dfrac{D}{2} - \dfrac{D^3}{12Bd_0}\tan(90-\phi)$

Note: The equation for calculation of hyperbola parameter ϕ' used in above tables is given in equation (61)

After giving the equations for the centre of buoyancy curve above, the metacentric curve equations for all three heel angle ranges in the first coordinate system quadrant are shown in the Table 4, below. The equations are given in the explicit form because their function argument is obtained using equations (8) and (9) for metacentre calculation, using formulas from the Table 2.

Table 4 Explicit metacentric curve equations for rectangle

Range	Heel Angles	Metacentric curve
I)	$0° \le \phi < \phi_1$	$y_M^2 = 32d/(9B^2)z_M^3 - 8/9\, z_M^2 + 2B^2/(27d)z_M - B^4/(243\cdot 2d)^2$
IIa)	$\phi_1 \le \phi < \phi_2$ $A \le A_T/2$	$z'^{\frac{2}{3}}_M - y'^{\frac{2}{3}}_M = \left(2\cdot\frac{2}{3}\sqrt{\frac{Bd}{\tan(\alpha/2)}}\right)^{\frac{2}{3}}$
IIb)	$\phi_1 \le \phi < \phi_2$ $A > A_T/2$	$z'^{\frac{2}{3}}_M - y'^{\frac{2}{3}}_M = \left(2\cdot\frac{2}{3}\frac{D-d}{D}\sqrt{\frac{B(D-d)}{\tan(\alpha/2)}}\right)^{\frac{2}{3}}$
III)	$\phi_2 \le \phi \le 90°$	$z'^2_M = 32Bd/(9D^3)y'^3_M - 8/9\, y'^2_M + 2D^3/(27Bd)y'_M - D^6/(243\cdot 2Bd)^2$

Finally, it can be seen from the tables above that all equations are obtained for arbitrary heel angles ϕ with input data about geometrical rectangular cross section dimensions: breadth B, height D, and initial hydrostatic immersion position defined by initial draught $d = d_0$, only. In that way, the goal of the paper is fulfilled completely.

3.6 Relation between global and rotated coordinate systems rotation heel angles

It is still left to define to relation between heel angles ϕ' in coordinate system $y' - z'$ initially rotated for angle ρ', used for hyperbola quadratic functions determination, and global coordinate system $y - z$ for which heel angles ϕ and centre of buoyancy and metacentric curves are defined. For that purpose, the new theorem is set here with an assumption that the tangent of observed centre of buoyancy curve is parallel to observed inclined waterline WL, i.e. they have the same heel angle ϕ, as shown in the **Fig. 6**, below.

Theorem 2: Parallelism of centre of buoyancy curve tangents and inclined waterlines

The tangent of some centre of buoyancy curve of general immersed geometry is parallel with its generating inclined waterline.

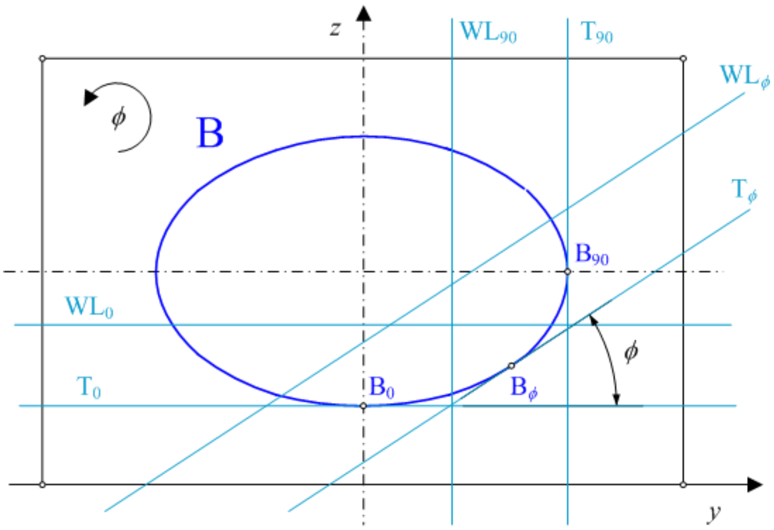

Fig. 6 Centre of buoyancy B-curve tangents parallel to inclined waterline

The assumption in the *Theorem 2* above is coming from the fact that the tangents T_0 and T_{90} to the centre of buoyancy B-curve for $\phi = 0°$ and $\phi = 90°$ are known to be parallel to their waterlines WL_0 and WL_{90}. Thus, the same is assumed for general heel angle ϕ of observed waterline WL_ϕ with tangent T_ϕ, as shown in the **Fig. 6**.

In this way, using *Theorem 2*, the hyperbola function, H, has minimum in rotated coordinate system $y-z$ for some arbitrary heel angle ϕ with

$$\min(dz/d\phi) \rightarrow dz/d\phi = 0 \tag{58}$$

Therefore, the parametric form of hyperbola must be used here, since the rotation for general heel angle ϕ has to be done in order to determine its corresponding ϕ' value with

$$z' = a'/\cos(\phi')$$

$$y' = 2/3b'\tan(\phi')$$

And here, it will be done for hyperbola cases, for Range II), since the parabola case for Range III) is already solved in literature, with following procedure starting with rotation of hyperbola for general heel angle ϕ, below.

$$z = z'\cos(\phi-\rho') + y'\sin(\phi-\rho') = a'/\cos(\phi')\cos(\phi-\rho') + 2/3b'\tan(\phi')\sin(\phi-\rho')$$
$$y = -z'\sin(\phi-\rho') + y'\cos(\phi-\rho') = -a'/\cos(\phi')\sin(\phi-\rho') + 2/3b'\tan(\phi')\cos(\phi-\rho')$$

Then

$$dz/d\phi = -a'/\cos(\phi')\sin(\phi-\rho') + 2/3b'\sin(\phi')/\cos(\phi')\cos(\phi-\rho')$$

After rearranging above and using the extreme condition in (58) one gets

$$a'\tan(\phi-\rho') - 2/3b'\sin(\phi') = 0$$

By substitution $\sin(\phi') = x$ the form of above is

$$a'\tan(\phi-\rho') = 2/3b'x \rightarrow$$

$$x = 3a'/(2b')\tan(\phi-\rho') \tag{59}$$

Finally, the relation between heel angles ϕ and ϕ' is

$$\phi' = \arcsin\left(\frac{3}{2}\frac{a'}{b'}\tan(\phi-\rho')\right) \tag{60}$$

For b' and a' shown for d', above equation in (60) becomes the simplest with

$$\phi' = \arcsin(\tan(\phi-\rho')) \tag{61}$$

And thus the relation between ϕ and ϕ' is obtained.

Since the heel angle ϕ and rotation angle ρ' are defined, its belonging hyperbola H parameter angle ϕ' can be determined using equations (60) and (61).

Above relation is then used for determination of the centre of buoyancy B-curve and metacentric M-curve points for arbitrary heel angles ϕ in the Range II) of global coordinate system $y-z$, in the Chapter 3, with examples given in the Chapter 4 of this paper.

4 Example - metacentric curves equations calculation

The example of the hydrostatic curves in this chapter are calculated for the rectangular cross section test pontoon from [26], with breadth B = 2.2 (m), height D = 1.54 (m), length L = 1 (m), as in the Part 1 of the paper, [26], in order to show metacentric curve cusp discontinuity existence in the hyperbola segment. There are two initial draughts

$d = d_0 = 0.3$ (m) and 1.2 (m) chosen for this purpose showing the situations for the Range IIa) with *Condition 1*: $A \leq A_T/2$ and Range IIb) with *Condition 1*: $A > A_T/2$, from (1).

In order to validate the calculation methods, the results of the calculations are checked for the angles on the joints of the heel angle segments, i.e. deck immersion/bottom emersions heel angles ϕ_1 and ϕ_2, and heel angle in the middle of the hyperbola segment, i.e. $\phi = 45°$, where the quadratic function B-curve values and metacentric locus M-curve can be checked for accuracy.

4.1 Rotated basic geometric shapes example

In this example, the calculation draught is $d = 0.3$ (m), with direct rotation of basic geometric shape with quadratic function as the equation of its centre of buoyancy B-curve.

The swallowtail discontinuity is shown in the Fig. 7, below, as expected for the case where $A/A_T \leq 1/2$.

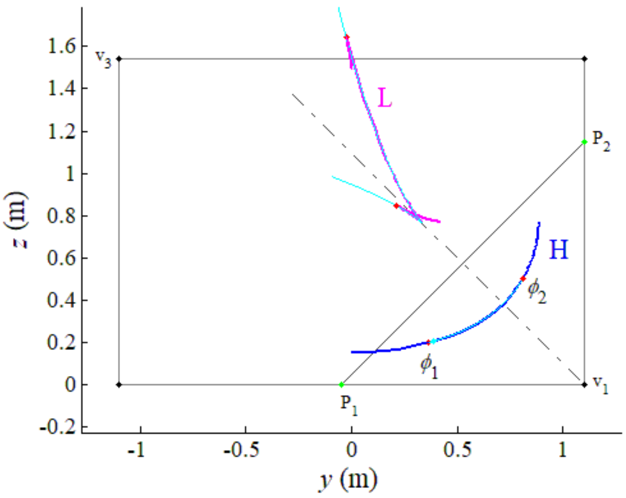

Fig. 7 Rotated basic triangle with hyperbola and its evolute for $A \leq A_T$

The marks "H" and "L" designate hyperbola and cuspidal Lamé curve, respectively. The exact formulas for deck immersion/bottom emersion heel angles ϕ_1 and ϕ_2 from Uršić, [26], are:

$$\phi_1 = \tan^{-1}(2d_0/B) = 15.25512°$$

$$\phi_2 = \tan^{-1}\left(D^2/(2Bd_0)\right) = 60.90029°$$

The equations for belonging centre of buoyancy B-curves for those heel angles ϕ_1 and ϕ_2, also from Uršić, [26], with corrected formulas for z are

$$\text{Range II)}\quad y = \frac{B}{2} - \frac{1}{3}\sqrt{2Bd_0/\tan\phi}\,,\quad z = \frac{1}{3}\sqrt{2Bd_0\tan\phi}$$

$$\text{Range III)}\quad y = \frac{B - Bd_0/D}{2} - \frac{D^3}{24Bd_0}\tan^2\phi\,, z = \frac{D}{2} - \frac{D^3}{12Bd_0}\tan\phi$$

The parametric equation for hyperbola H in the Range IIa) using equations (41) and (42) is

$$z' = a'/\cos(\phi') \rightarrow z' = 0.541603/\cos(\phi')$$

$$y' = 2/3b'\tan(\phi') \rightarrow 0.541603\tan(\phi')$$

The exact equation for hyperbola H in the Range IIa) using equation (16) is then:

$$4/9\cdot 0.812404^2 z'^2 - 0.541603^2 y'^2 = 4/9\cdot 0.812404^2 \cdot 0.541603^2$$
$$0.541603 z'^2 - 0.541603^2 y'^2 = 0.0860\dot{4}^2$$

$$z' = \sqrt{0.29\dot{3} + y'^2}$$

And equation for belonging cuspidal Lamé curve L for Range IIb) is:

$$z' = \frac{1}{a'}\left[\left(\frac{2b'}{3}y'\right)^{\frac{2}{3}} + \left(\left(\frac{2b'}{3}\right)^2 + a'^2\right)^{\frac{2}{3}}\right]^{\frac{3}{2}} \rightarrow (a'z')^{\frac{2}{3}} - \left(\frac{2b'}{3}y'\right)^{\frac{2}{3}} = \left(\frac{4b'^2}{9} + a'^2\right)^{\frac{2}{3}}$$

$$(0.541603 z')^{\frac{2}{3}} - (2/3\cdot 0.812404 y')^{\frac{2}{3}} = (4/9\cdot 0.812404^2 + 0.541603^2)^{\frac{2}{3}}$$

$$\rightarrow (0.541603 z')^{\frac{2}{3}} - (0.541603 y')^{\frac{2}{3}} = (2\cdot 0.541603^2)^{\frac{2}{3}}$$

$$z'^{\frac{2}{3}} - y'^{\frac{2}{3}} = (2\cdot 0.541603)^{\frac{2}{3}}$$

After all above equations and parameters are obtained, the results regarding centre of buoyancy and metacentric curve can be checked for three angles, deck immersion/bottom emersions heel angles ϕ_1 and ϕ_2 and $\phi = 45°$, as shown in the Table 5, below.

Table 5 The results of centre of buoyancy and metacentric curves for chosen heel angles

Heel angle	$\phi_1 = 15.25512°$		$\phi_2 = 60.90029°$		$\phi = 45°$	
Hyperbola parameter	$\phi'_1 = -34.8499046°$		$\phi'_2 = 16.5506556°$		$\phi' = 0°$	
Hyperbola ($y' - z'$)	y' (m)	z' (m)	y' (m)	z' (m)	y' (m)	z' (m)
X'_B	-0.377123	0.659966	0.160951	0.565012	0	1.10111
B-curve	y (m)	z (m)	y (m)	z (m)	y (m)	z (m)
Parabola Range I)	0.36674	0.200014	-	-	-	-
Hyperbola	0.3667	0.200019	0.814286	0.513333	0.717029	0.382971
Parabola Range III)	-	-	0.814286	0.513333	-	-
X_B	$0.3\dot{6}$	0.2	0.814286	$0.51\dot{3}$	0.717029	0.382971
M-curve	y (m)	z (m)	y (m)	z (m)	y (m)	z (m)
Semi-cubical parabola Range I)	-0.02727	$1.6\dot{4}$	-	-	-	-
Cuspidal Lamé curve	-0.02727	1.6444	0.210284	0.849512	0.333994	0.766005
Semi-cubical parabola Range III)	-	-	0.210284	0.849512	-	-
X_M	$-0.0\dot{2}\dot{7}$	$1.6\dot{4}$	0.210284	0.849512	0.334058	0.765942

Note: The relation for calculation of hyperbola angles ϕ', for known heel angles ϕ, is determined in the equations (60) and (61).

In order to obtain equations in global $y - z$ coordinate system, the formulas in hyperbola section of the centre of buoyancy curve should be set to origin, (13). Of course, for the calculation to be correct, the values obtained for parabolas and hyperbola must be the same at the angles ϕ_1 and ϕ_2 and with the values for $\phi = 45°$, as confirmed in above in the Table 5.

And now the example of scaling using hydrostatic area complement method will be examined in the next subchapter.

4.2 Scaling using hydrostatic cross section area complement method

In order to show the application of scaling using direct hydrostatic cross section area complement method, the figure showing calculation for above example is shown in the Fig. 8, below, for initial draught d = 1.2 (m).

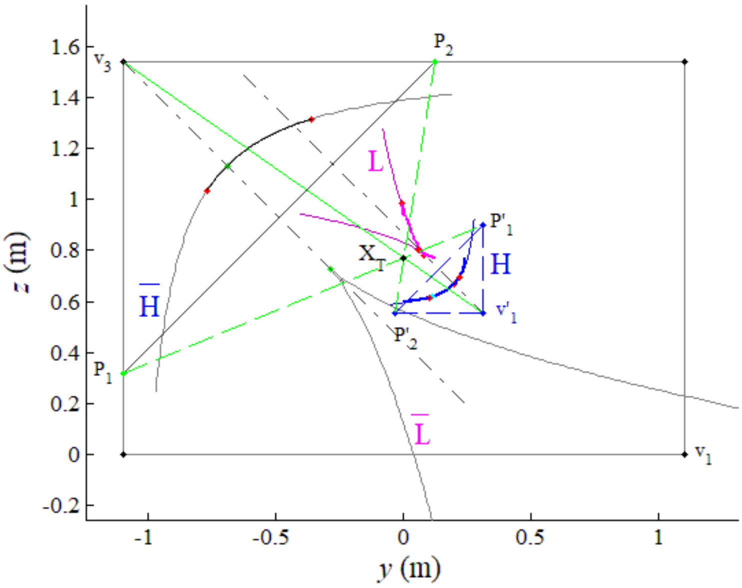

Fig. 8 Scaling method using Hydrostatic cross section area complement

The description of the Fig. 8 above is following: The marks "H" and "L", and "$\overline{\mathrm{H}}$" and "$\overline{\mathrm{L}}$" designate hyperbola and cuspidal Lamé curve equations for initial B and M-curves and complementary $\overline{\mathrm{B}}$ and $\overline{\mathrm{M}}$ -curves, respectively. The green lines represent the scaling of complementary cross section area defined by triangle $\Delta(P_1, P_2, v_3)$ to triangle $\Delta(P'_1, P'_2, v'_1)$ about scaling point centre X_T, thus obtaining generator shape for construction of initial B and M-curves of hyperbola H and cuspidal Lamé curve L. Blue curve represents B-curve with bold hyperbolic part H, and magenta colour designates M-curve with bold cuspidal Lamé curve part L.

From the equations for triangular complementary cross section area semi-axes in (30) and (31) their values can be calculated as

$$\bar{b} = \sqrt{2.2(1.54 - 1.2)\tan(90/2)} = 0.86487$$

$$\bar{a} = 2/3\sqrt{2.2(1.54 - 1.2)/\tan(90/2)} = 0.57658$$

The parametric equation for hyperbola $\overline{\mathrm{H}}$ using equations (47) and (48) is then:

$$\bar{z} = \bar{a}/\cos(\bar{\phi}) \rightarrow \bar{z} = 0.57658/\cos(\bar{\phi})$$

$$\bar{y} = 2/3 \cdot \bar{a}\tan(\bar{\phi}) \rightarrow \bar{y} = 2/3 \cdot 0.86487\tan(\bar{\phi}) \rightarrow \bar{y} = 0.57658\tan(\bar{\phi})$$

And the explicit equation for belonging cuspidal Lamé curve $\overline{\mathrm{L}}$ is:

$$\bar{z}=\frac{1}{\bar{a}}\left[\left(\frac{2\bar{b}}{3}\bar{y}\right)^{\frac{2}{3}}+\left(\bar{a}^{2}+\left(\frac{2\bar{b}}{3}\right)^{2}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}\rightarrow\left(\bar{a}\bar{z}\right)^{\frac{2}{3}}-\left(\frac{2\bar{b}}{3}\bar{y}\right)^{\frac{2}{3}}=\left(\frac{4\bar{b}^{2}}{9}+\bar{a}^{2}\right)^{\frac{2}{3}}$$

$$\left(0.57658\bar{z}\right)^{\frac{2}{3}}-\left(\frac{2\cdot 0.86487}{3}\bar{y}\right)^{\frac{2}{3}}=\left(\frac{4\cdot 0.86487^{2}}{9}+0.57658^{2}\right)^{\frac{2}{3}}$$

$$\rightarrow\left(0.57658\bar{z}\right)^{\frac{2}{3}}-\left(0.57658\bar{y}\right)^{\frac{2}{3}}=\left(2\cdot 0.57658^{2}\right)^{\frac{2}{3}}$$

$$\bar{z}^{\frac{2}{3}}-\bar{y}^{\frac{2}{3}}=\left(2\cdot 0.57658\right)^{\frac{2}{3}}$$

The semi-axes *b'* and *a'* then can be determined using equation (19) for *k* ratio with

$$k=-\left(A_T-A\right)/A=-\left(BD-Bd\right)/\left(Bd\right)=-\left(D-d\right)/d=-\left(1.54-1.2\right)/1.2=-0.28\dot{3}$$

$$b'=|k|\cdot\bar{b}=0.28\dot{3}\cdot 0.86487\rightarrow b'=0.245046$$

$$a'=|k|\cdot\bar{a}=0.28\dot{3}\cdot 0.57658\rightarrow b'=0.16336$$

The parametric equation for hyperbola H in the Range IIb) using equations (47) and (48) is

$$z'=\frac{2}{3}\frac{2.2\cdot 1.54-2.2\cdot 1.2}{2.2\cdot 1.2}\sqrt{\frac{2.2\cdot 1.2}{\tan 45^{\circ}}}\frac{1}{\cos\phi'}\rightarrow z'=\frac{0.306908}{\cos\phi'}$$

$$y'=\frac{2}{3}\frac{2.2\cdot 1.54-2.2\cdot 1.2}{2.2\cdot 1.2}\sqrt{2.2\cdot 1.2/\tan 45^{\circ}}\tan\phi'\rightarrow y'=0.306908\cdot\tan\phi'$$

The explicit equation for hyperbola H in the Range IIb) using equation (16) is then:

$$\frac{4\cdot 0.245046^{2}}{9}z'^{2}-0.16336^{2}y'^{2}=\frac{4\cdot 0.245046^{2}\cdot 0.16336^{2}}{9}$$

$$0.16336\,z'^{2}-0.16336^{2}y'^{2}=0.026688^{2}$$

$$z'=\sqrt{0.026688+y'^{2}}$$

And belonging equation for cuspidal Lamé curve L for Range IIb) is:

$$z'=\frac{1}{a'}\left[\left(\frac{2b'}{3}y'\right)^{\frac{2}{3}}+\left(\left(\frac{2b'}{3}\right)^{2}+a'^{2}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}\rightarrow\left(a'z'\right)^{\frac{2}{3}}-\left(\frac{2b'}{3}y'\right)^{\frac{2}{3}}=\left(\frac{4b'^{2}}{9}+a'^{2}\right)^{\frac{2}{3}}$$

$$\left(0.16336z'\right)^{\frac{2}{3}}-\left(\frac{2\cdot 0.245046}{3}y'\right)^{\frac{2}{3}}=\left(\frac{4\cdot 0.245046^{2}}{9}+0.16336^{2}\right)^{\frac{2}{3}}$$

$$\rightarrow\left(0.16336z'\right)^{\frac{2}{3}}-\left(0.16336y'\right)^{\frac{2}{3}}=\left(2\cdot 0.16336^{2}\right)^{\frac{2}{3}}$$

$$z'^{\frac{2}{3}}-y'^{\frac{2}{3}}=\left(2\cdot 0.16336\right)^{\frac{2}{3}}$$

Finally, the position of isosceles triangle, determined using scaling of complementary triangle with points $P_1(y_1, z_1)$, $P_2(y_2, z_2)$ and $v_3(-B/2, D)$, has to be determined for its points $P'_1(y'_1, z'_1)$, $P'_2(y'_2, z'_2)$ and $v'_1(y'_{v1}, z'_{v1})$, using vector form of scaling method from equation (22) for centre point $X_T = (0, D/2)$. The results are shown in the Table 6, below.

Table 6 The position of scaled isosceles triangle

$P'_1(P_1)$		$P'_2(P_2)$		$v'_1(v_3)$	
$y_1 = -B/2$	$z_1 = 0.316889$	$y_2 = 0.12311$	$z_2 = 1.54$	$y_{v3} = -B/2$	$z_{v3} = 1.54$
$y_T - k(y_1 - y_T)$	$z_T - k(z_1 - z_T)$	$y_T - k(y_2 - y_T)$	$z_T - k(z_1 - z_T)$	$y_T - k(y_{v3} - y_T)$	$z_T - k(z_{v3} - z_T)$
0-0.283(-1.1-0)	0.77-0.283(0.316889-0.77)	0-0.283(0.12311-0)	0.77-0.283·0.77	0-0.283(-1.1-0)	0.77-0.283·0.77
0.311666	0.89838145	-0.034881167	0.5518333	0.311666	0.5518333

From the scaled isosceles triangle in the Table 6, their semi-axes values can be rechecked with the b' determination using the distance between points $P'_1(y'_1, z'_1)$ and $P'_2(y'_2, z'_2)$ with

$$b' = \sqrt{(y'_1 - y'_2)^2 + (z'_1 - z'_2)^2} / 2$$

$$b' = \sqrt{(0.311666 - 0.03488117)^2 + (0.89838145 - 0.5518333)^2} / 2 = 0.245046$$

And it can be concluded that semi-axes results are accurate and equal to b' value previously calculated using k ratio.

It is still needed to check the accuracy of above calculation method and here it will be done for three centre of buoyancy and metacentric curve points for the first and the second deck immersion/bottom emersion heel angles ϕ_1 and ϕ_2, and $\phi = 45°$ where hyperbola has minimum, in the Table 7, below. The exact formulas for heel angles ϕ_1 and ϕ_2 from Uršić, [26], are:

$$\phi_1 = \tan^{-1}\left(2(D - d_0)/B\right) = 17.1759°$$

$$\phi_2 = \tan^{-1}\left(D^2/\left(2B(D - d_0)\right)\right) = 57.7564°$$

The equations for belonging centre of buoyancy B-curves for those heel angles ϕ_1 and ϕ_2 also from Uršić, [26], with corrected formula for z in Range III) are

$$y = \frac{B - Bd_0 / D}{2} - \frac{D^3}{24Bd_0}\tan^2\phi,\ z = \frac{D}{2} - \frac{D^3}{12Bd_0}\tan\phi$$

The equations in the Range II) can be obtained from the parametric hyperbola equations (41) and (42), translated in origin of $y - z$ coordinate system and rotated for heel angles ϕ_1, ϕ_2, and $\phi = 45°$, using affine transformation equation in (12) and an equation for relation between initial heel angles ϕ and hyperbola parameter angle ϕ' in (61).

With this final check of the results from the Table 7, and the results from the Table 6, the equations for Range II) segment of centre of buoyancy B-curve are given in this paper, with hyperbola and cuspidal Lamé curve equations for both situations regarding the *Condition 1* cases $A/A_T \leq 1/2$ and $A/A_T > ½$, in (1).

Table 7 The results for centre of buoyancy and metacentric curves for chosen heel angles

Heel angle	ϕ_1 = 17.1759°		ϕ_2 = 57.7564°		ϕ = 45°	
Hyperbola parameter	ϕ'_1 = -31.85544°		ϕ'_2 = 13.08488°		ϕ' = 0°	
Hyperbola ($y' - z'$)	y' (m)	z' (m)	y' (m)	z' (m)	y' (m)	z' (m)
X'_{B}	-0.10139	0.192297	0.037926	0.167687	0	0.163333
B-curve	y (m)	z (m)	y (m)	z (m)	y (m)	z (m)
Parabola Range I)	0.103890	0.616056	-	-	-	-
Hyperbola	0.103893	0.616057	0.21992	0.697278	0.196151	0.667349
Parabola Range III)	-	-	0.21992	0.697278	-	-
X_{B}	0.103889	0.616056	0.21992	0.697278	0.196151	0.667349
M-curve	y (m)	z (m)	y (m)	z (m)	y (m)	z (m)
Semi-cubical parab. Range I)	-0.009925	0.984278	-	-	-	-
Cuspidal Lamé curve	-0.009925	0.984278	0.0587614	0.798936	0.080635	0.782865
Semi-cubical parab. Range III)	-	-	0.0587615	0.798937	-	-
X_{M}	-0.009925	0.984278	0.0587615	0.798937	0.080634	0.782865

Except hyperbola part, all other equations for centre of buoyancy and metacentric curves are given for hell angle ranges I) and III), thus proving *Definition 1* from the Part 1 of the paper, [26], also. Moreover, the part of *Definition 2* , from the Part 1 of the paper, [26], regarding the position of the extreme of hyperbola part of centre of buoyancy B-curve on the line $y = \pm x$ or ±45° is proved here also using above described scaling using hydrostatic complementary area and rotated basic geometry shapes methods.

Thus, the *Theorem* 2 is approved, also, confirming parallelism of inclined waterlines and centre of buoyancy B-curve tangents.

4.3 Overall equations for the examples

Finally, the exact equations for centre of buoyancy and metacentric curve can be given for the examples of immersed rectangle from this chapter, i.e. draughts d = 0.3 (m) and 1.2 (m), are shown in the Tables 8 and 9, below, using formulas from the Tables 3 and 4.

Table 8 Centre of buoyancy and metacentric curve equations for draught d = 0.3 (m)

Range	Heel Angles	Centre of Buoyancy Curve	
I)	$0° \le \phi < \phi_1$	$y_{\mathrm{B}} = 1.3\dot{4}\tan\phi$	$z_{\mathrm{B}} = 0.67\dot{2}\tan^2\phi + 0.15$
IIa)	$\phi_1 \le \phi < \phi_2$	$y'_{\mathrm{B}} = 0.541603\tan\phi'$	$z'_{\mathrm{B}} = 0.541603/\cos\phi'$
III)	$\phi_2 \le \phi \le 90°$	$y_{\mathrm{B}} = 0.885714 - 0.230572\tan^2\phi'$	$z_{\mathrm{B}} = 0.77 - 0.046144\tan\phi'$
Range	Heel Angles	Metacentric Curve	
I)	$0° \le \phi < \phi_1$	$y_{\mathrm{M}}^2 = 0.220386 z_{\mathrm{M}}^3 - 0.\dot{8} z_{\mathrm{M}}^2 + 1.195062 z_{\mathrm{M}} - 0.001102$	
IIa)	$\phi_1 \le \phi < \phi_2$	$z'^{\frac{2}{3}}_{\mathrm{M}} - y'^{\frac{2}{3}}_{\mathrm{M}} = (2 \cdot 0.541603)^{\frac{2}{3}}$	
III)	$\phi_2 \le \phi \le 90°$	$z'^2_{\mathrm{M}} = 0.642524 y'^3_{\mathrm{M}} - 0.\dot{8} y'^2_{\mathrm{M}} + 0.409906 y'_{\mathrm{M}} - 0.0001296$	

The hydrostatic properties values for d = 0.3 (m) are checked using Ban's L_1 norm polynomial radial basis functions, PRBFs, [31], as shown in the Fig. 9, below, with graphical representation, where the calculations prove above results.

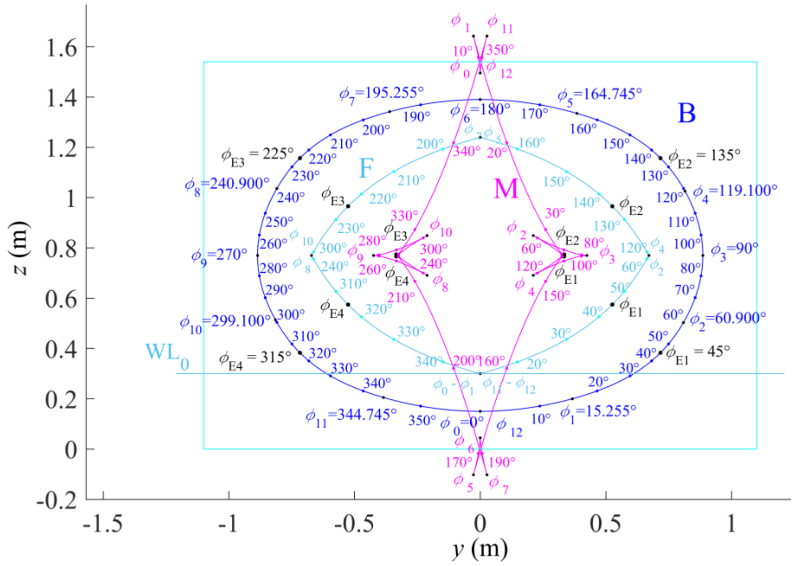

Fig. 9 Hydrostatic properties for rectangular cross section pontoon, for d = 0.3 (m)

Finally, the calculation results for draught d = 1.2 (m), are given in the Table 9, below, with graphical results shown in the Part 1 of the paper, [26], thus proving obtained results.

Table 9 Centre of buoyancy and metacentric curve equations for draught d = 1.2 (m)

Range	Heel Angles	Centre of Buoyancy Curve	
I)	$0° \le \phi < \phi_1$	$y_{\mathrm{B}} = 0.336\dot{1}\tan\phi$	$z_{\mathrm{B}} = 0.1680\dot{5}\tan^2\phi + 0.6$
IIb)	$\phi_1 \le \phi < \phi_2$	$y'_{\mathrm{B}} = 0.306908\tan\phi'$	$z'_{\mathrm{B}} = 0.306908/\cos\phi'$
III)	$\phi_2 \le \phi \le 90°$	$y_{\mathrm{B}} = 0.242857 - 0.05764\tan^2\phi'$	$z_{\mathrm{B}} = 0.77 - 0.115268\tan\phi'$
Range	Heel Angles	Metacentric Curve	
I)	$0° \le \phi < \phi_1$	$y_{\mathrm{M}}^2 = 0.881543 z_{\mathrm{M}}^3 - 0.\dot{8} z_{\mathrm{M}}^2 + 0.298765 z_{\mathrm{M}} - 6.887\cdot 10^{-5}$	
IIb)	$\phi_1 \le \phi < \phi_2$	$z'^{\frac{2}{3}}_{\mathrm{M}} - y'^{\frac{2}{3}}_{\mathrm{M}} = (2\cdot 0.16336)^{\frac{2}{3}}$	
III)	$\phi_2 \le \phi \le 90°$	$z'^2_{\mathrm{M}} = 2.570095 y'^3_{\mathrm{M}} - 0.\dot{8} y'^2_{\mathrm{M}} + 0.102476 y'_{\mathrm{M}} - 8.103\cdot 10^{-6}$	

In this way, all centre of buoyancy B-curve and metacentric locus M-curve segment equations are given, with explicit and parametric quadratic functions for parabolas and hyperbolas, and explicit semi-cubical and cuspidal Lamé curves equations.

5 Conclusion

The centre of buoyancy and metacentric curve equations for rectangular cross section shape are given using quadratic functions, in this Part 2 of the paper, after upper and lower non-dimensional bounds for swallowtail discontinuity existence are given in the first Part 1 of it. That is, the parabola and hyperbola equations for the centre of buoyancy curve segments are given in the explicit and parametric form, not defined so far, thus proving hyperbola segment existence of centre of buoyancy curve for rectangular cross section, also. Except that, the explicit equations for all metacentric curve segments in the first coordinate system quadrant are given, for semi-cubical parabolas and Lamé curves with 2/3 exponent and negative sign. In order to obtain belonging quadratic functions for description of above curves, two new methods are given: rotated basic geometrical shapes and scaling method using hydrostatic cross section area complement. Both methods use basic geometrical shapes as generators for quadratic functions to be applicable for the centre of buoyancy curve description, thus enabling giving their direct equation. They show also, that additional extreme cusp discontinuities of metacentric curve for rectangular cross section floating body exists for heel angles equal 45 degrees, i.e. in the symmetry of the rectangle vertex angles, thus proving the conclusions from the Part 1 and Part 2 of the paper, for swallowtail type cusp discontinuities existence.

Additionally, in order to obtain above general equations for rotated basic shapes, two new theorems are given in the paper, with one of them showing that the centre of buoyancy B-curve tangent is parallel with observed waterline, therefore representing important hydrostatic property that should be further examined. The other theorem is related to the hydrostatic cross section area complement scaling method that uses mathematical homothety to determine the generating isosceles triangle shape for hyperbola quadratic function when pentagonal shape occurs, with simple scale ratio determination using ratio of initial, immersed and complementary, emerged cross section area of rectangle.

Since all the equations of the centre of buoyancy and metacentric curve segments for the rectangular cross section are known now, its hydrostatic kinematics can be re-examined in detail, and that will be the future research topic of the author of this paper.

After above is examined for the regular rectangular shape of a floating body, in the future author's work it will be investigated for other regular shapes, as well as for the cross-sectional shapes of actual ship hull forms. Except two-dimensional problems, there are three-dimensional regular bodies to be examined using quadratic functions, and that will be done in the future author's work, also, with rectangular prismatic pontoon to be examined first.

Nomenclature

α – rectangle vertex angle, (°),

β – quadratic function parameter,

ϕ_1 – heel angle, (°),

ϕ_i – cusp discontinuity heel angle, (°),

ϕ_1, ϕ_2 – deck immersion/bottom emersion heel angles, (°),

ρ – geometrical shape rotation angle, (°),

∇ – volume displacement of the floating body, (m^3),

b, a – quadratic function horizontal and vertical semi-axes, (m),

b', a' – quadratic function horizontal and vertical semi-axes in rotated coordinate system, (m),

$\bar{b}$, $\bar{a}$ – quadratic function horizontal and vertical semi-axes for hydrostatic complement, (m),

b_i – breadth in i-th heel angles range, (m),

d – draught, (m),

d_0 – initial draught, (m),

d_i – draught in i-th heel angles range, (m),

k – scaling ratio,

p – parabola parameter,

v – rectangle vertex, (m),

y, z – general rectangle coordinates,

y_B, z_B – centre of buoyancy curve coordinates, (m),

y_M, z_M – metacentric curve coordinates, (m),

A – cross section area, (m^2),

A' – cross section area of rotated basic geometrical shape, (m^2),
$\overline{A}$ – cross section area of rotated basic geometrical shape for hydrostatic complement, (m^2),
A_0 – initial immersed cross section area, (m^2),
A_C – complementary cross section area, (m^2),
A_E – emerged cross section area of rectangle above WL, (m^2),
A_I – immersed cross section area of rectangle below WL, (m^2),
A_T – total cross section area of rectangle, (m),
B – centre of buoyancy designation,
B – breadth of rectangle, (m),
D – draught of rectangle, (m),
E – extreme cusp discontinuity angle, (°),
F – centre of waterline designation,
G – geometrical feature,
H – hyperbola curve,
$I(\phi)$ – moment of inertia of actual waterline depending on angle of heel ϕ, (m^4),
L – Lamé curve with 2/3 exponent and negative sign (cuspidal Lamé curve),
M – metacentre designation,
$\overline{\mathrm{MB}}$ – transversal metacentric radius, (m),
P – point vector of geometry,
R_ρ – rotation of geometrical shape for angle ρ,
T – tangent,
T_v – translation of geometrical shape to rectangle vertex v,
WL – waterline,
$\mathrm{WL}_{i\text{-}i+1}$ – waterline between two cusp discontinuity heel angles ϕ_i and ϕ_{i+1},
X_T – centroid of geometrical shape; scaling centre point.

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Dario Ban, Dario.Ban@fesb.hr
University of Split
Faculty of Electrical Engineering, Mechanical Engineering and Naval Architecture, 21000 Split; Croatia